LOGISTIC POPULATION MODEL $dP/dt = kP(1 - P/N)$ SEPARABLE FIRST ORDER EQUATION $dy/dt = g(t)h(y)$ AUTONOMOUS a de that does not depend on time EXISTENCE THEOREM if $f(t, y)$ is continuous in an open rectangle R, and $(t_0, y_0) \in R$, then $\exists \epsilon > 0, y(t)$ where $y(t)$ is defined for $t_0 - \epsilon < t < t_0 + \epsilon$ and $dy/dt = f(t, y), y(t_0) = y_0$ UNIQUENESS THEOREM if $\partial f/\partial y$ is also continuous in R, and both $y_1(t)$ and $y_2(t)$ satisfy $dy/dt = f(t, y), y(t_0) = y_0$ for all t where $t_0 - \epsilon < t < t_0 + \epsilon$ for some $\epsilon > 0$, then $y_1(t) = y_2(t)$ HOMOGENEOUS LINEAR $dy/dt = a(t)y$. sol'n $y = ke^{\int a dt}$ for any k. also called "unforced." LINEAR FIRST ORDER $dy/dt = a(t)y + b(t)$. solution is $ky_1 + y_2$ where y_2 is any particular solution, and y_1 solves homogeneous $y' = a(t)y$ GUESSING LINEAR SYSTEMS if forcing term is exponential, guess ke^t or kte^t. if forcing term is $\sin \alpha t$, guess $k_1 \sin \alpha t + k_2 \cos \alpha t$. if forcing term is polynomial, guess a polynomial one term higher. INTEGRATING FACTOR if $dy/dt + g(t)y = b(t)$ then let $u(t) = e^{\int g(t)dt}$ and $y = 1/u(t) \int u(t)b(t)dt$ is a sol'n AUTONOMOUS SYSTEM is $dY/dt = F(Y)$ where $Y \in R^n$, $F : R^n \to R^n$ DECOUPLE a system is decoupled when subsets of variables depend only on themselves. Those variables can be solved separately. LINEAR HIGHER ORDER HOMOGENEOUS CONSTANT COEFF eqn is $a_n d^n y/dt^n + ... + a_0 y = 0$. guess e^{st}. get characteristic eqn $a_n s^n + ... + a_0 s^0 = 0$, solve for s HARMONIC OSCILLATOR $md^2y/dt^2 + bdy/dt + ky = 0$ where b is a damping term EXISTENCE AND UNIQUENESS FOR SYSTEMS let $dY/dt = F(t, Y)$. If F is continuously diff. then for t_0, Y_0 $\exists \epsilon > 0, Y(t)$ defined on $t_0 - \epsilon < t < t_0 + \epsilon$ where $dY/dt = F(t, Y)$ and $Y(t_0) = Y_0$, and this solution is unique LINEAR SYSTEM $dY/dt = AY$ where Y is a vector and A is a matrix of constants. if $Y_1(t), Y_2(t)$ are solutions, then so is $k_1Y_1(t) + k_2Y_2(t)$ STRAIGHT LINE SOLNS OF LINEAR SYSTEM to solve find eigenvalues/eigenvectors λ_i, v_i of A (i.e. solve $det(A - I\lambda) = 0$ for λ) and straight line solns are $Y = ke^{\lambda_i t}v_i$ for arbitrary k. if A is 2x2 with 2 real eigenvalues then $Y = k_1 e^{\lambda_1 t} v_1 + k_2 e^{\lambda_2 t} v_2$ COMPLEX EIGENVALUE SOLNS OF LINEAR SYSTEM in a linear system if the eigenvalues are complex then Re Y and Im Y are both solns, where Y is found as in the straight line solns. If the system has 2 variables and 2 complex eigenvalues $\alpha \pm i\beta$, then if $\alpha < 0$ the origin is a spiral sink, if $\alpha > 0$ the origin is a spiral source, and if $\alpha = 0$ the origin is a center REPEATED EIGENVALUES if a linear system $Y' = AY$ has a repeated real eigenvalue λ and initial condition V_0 then $e^{\lambda t}V_0 + te^{\lambda t}(A - \lambda I)V_0$ is a solution. ZERO EIGENVALUE if there is a 0 eigenvalue in a 2 var system, solutions move in a straight line to that eigenvector, which generates a line of equilibrium points GENERAL LINEARITY PRINCIPLE for a linear DE of higher order is: if $y_p(t)$ solves the nonhomogeneous eqn and $y_h(t)$ solves the homogeneous equation, then $y_h(t) + y_p(t)$ solves the nonhomogeneous equation. If $y_p(t)$ and $y_q(t)$ solve the nonhomogeneous eqn, then $y_p(t) - y_q(t)$ solves the homogeneous eqn NON-HOMOGENEOUS LINEAR SYSTEMS OF HIGHER ORDER can be solved by guessing a solution as in the first-order case. CHANGE OF VARIABLES in a d.e. constitutes making a substitution $u = f(x, t)$ and calculating du/dt in terms of dy/dt. It can be used for instance to eliminate t on the rhs (making the eqn autonomous) BERNOULLI EQUATION $dy/dt = r(t)y + a(t)y^n$. solve by letting $z = y^{1-n}$, getting $dz/dt = (1-n)y^{-n}dy/dt = (1-n)y^{-n}(r(t)y + a(t)y^n) = (1-n)(r(t)z + a(t))$ which is linear and can be solved RICATTI EQUATION $dy/dt = r(t) + a(t)y + b(t)y^2$ where $r(t) = f(t, 0)$, $a(t) = (\partial f/\partial y)(t, 0)$, $b(t) = ((\partial^2 f/\partial^2 y)/2!)(t, 0)$. it arises as the second-order Taylor approximation to $dy/dt = f(t, y)$. to solve, first somehow find a solution y_1. then let $w = y - y_1$, $dw/dt = dy/dt - dy_1/dt = a(t)(y - y_1) + b(t)(y^2 - y_1^2) = a(t)w + b(t)(y - y_1)(y - y_1 + 2y_1) = (a(t) + 2y_1 b(t))w + b(t)w^2$ which is a Bernoulli equation and can be solved by that method. ULTIMATE GUESS guess that the solution is a Taylor series. write the d.e. in terms of finite sums, replacing dy/dt with the formal derivative of the power series. Equate coefficients on either side, getting an infinite family of linear equations to solve for the coefficients. HERMITE EQUATION $d^2y/dt - 2tdy/dt + 2py = 0$ using the ultimate guess method gives a power series such that each term beyond a certain order n shares a factor $(p - k)$, for k a non-negative integer. thus if p is a non-negative integer, all terms of the power series vanish beyond a certain order, so you get a solution which is a polynomial $H_p(t)$. HERMITE POLYNOMIALS the first few $H_p(t)$ are $H_0(t) = 1$, $H_1(t) = t$, $H_2(t) = 1 - 2t^2$, and more can be generated COMPLEXIFICATION replace the forcing term $\cos \omega t$ with $e^{i\omega t}$, then the real part of the solution solves the eqn CONSERVED QUANTITY of a system is a function $H(Y)$ that is constant along any solution curve of the system HAMILTONIAN SYSTEM is a system in x, y such that there exists $H(x, y)$ where $dx/dt = \partial H/\partial y$ and $dy/dt = -\partial H/\partial x$. H is the Hamiltonian function for the system. H is a conserved quantity, therefore solution curves lie along the level curves of H. H is also called the energy function. DETERMINE IF SYSTEM IS HAMILTONIAN by checking whether $\partial f/\partial x = \partial^2 H/\partial x\partial y = \partial^2 H/\partial y\partial x = -\partial g/\partial y$ (alternately, $\nabla(f(x, y), g(x, y)) = 0$) FINDING HAMILTONIAN $H(x, y) = \int f(x, y)dy + \phi_1(x) = -\int g(x, y)dx + \phi_2(y)$ THEOREM a Hamiltonian system has no sinks or sources, only saddles and centers. If an equilibrium point in a Hamiltonian system has eigenvalues that are imaginary with 0 real part, it is a center. LYAPUNOV FUNCTION is $L(x, y)$ satisfying $d/dt\ L(x(t), y(t)) \leq 0$ for solution curves of the system, with strict inequality except for a discrete set of t's GRADIENT SYSTEM is a system where $dY/dt = \nabla G$ for some function G. $-G$ is a Lyapunov function for a gradient system. Gradient systems have no spiral sinks, sources, or centers. FOURIER TRANSFORM of $y(t)$ is Re $\int_{-\infty}^{\infty} y(t)e^{-zt}dt = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt$ LAPLACE TRANSFORM of the function $y(t)$ is $Y(s) = \mathcal{L}(y) = \int_0^{\infty} y(t)e^{-st}dt$ defined on all s where this integral converges. To solve a de with the Laplace transform, take $\mathcal{L}$ of both sides, solve for $\mathcal{L}[y]$, and take $\mathcal{L}^{-1}$ of both sides. HEAVISIDE FUNCTION $u_a(t) = 0$ if $t < a$, $u_a(t) = 1$ if $t \geq a$ DIRAC DELTA FUNCTION $\delta_a(t)$ is a "function" with a spike of infinite height and unit area at $t = a$, and 0 elsewhere LAPLACE TRANSFORM RULES $\mathcal{L}[f + g] = \mathcal{L}[f] + \mathcal{L}[g]$, $\mathcal{L}[cf] = c\mathcal{L}[f]$ if c is constant, $\mathcal{L}[dy/dt] = s\mathcal{L}[y] - y(0)$, $\mathcal{L}[u_a(t)y(t - a)] = e^{-as}\mathcal{L}[y]$, $\mathcal{L}[e^{at}y(t)] = \mathcal{L}[y(t)](s - a)$ INVERSE LAPLACE TRANSFORM $\mathcal{L}^{-1}(Y)$ is also linear. There is no explicit form, you have to rewrite Y into a form you recognize, often using partial fraction decomposition. INVERSE LAPLACE TRANSFORM RULES take the inverse of the laplace transform rules CONVOLUTION with $f * g(t) = \int_0^t f(t - u)g(u)du$ PRODUCT RULE FOR INVERSE LAPLACE TRANSFORM $\mathcal{L}^{-1}[FG] = \mathcal{F} * \mathcal{G}$. Usually using this rule yields tedious integrals, however. COMMON LAPLACE TRANSFORMS $\mathcal{L}[e^{at}] = 1/(s - a)$ for $s > a$, $\mathcal{L}[\sin \omega t] = \omega/(s^2 + \omega^2)$, $\mathcal{L}[e^{at} \sin \omega t] = \omega/((s - a)^2 + \omega^2)$, $\mathcal{L}[t \sin \omega t] = 2\omega s/(s^2 + \omega^2)^2$, $\mathcal{L}[u_a(t)] = e^{-as}/s$ for $s > 0$, $\mathcal{L}[t^n] = n!/s^{n+1}$ for $s > 0$, $\mathcal{L}[\cos \omega t] = s/(s^2 + \omega^2)$, $\mathcal{L}[e^{at} \cos \omega t] = (s - a)/((s - a)^2 + \omega^2)$, $\mathcal{L}[t \cos \omega t] = (s^2 - \omega^2)/(s^2 + \omega^2)^2$, $\mathcal{L}[\delta_a(t)] = e^{-as}$ QUALITATIVE examine the slope field to find asymptotes and approximate solution curves PHASE LINE for an autonomous eqn is a vertical line with a dot at every constant solution (equilibrium point) and an arrow between dots pointing in the direction of solutions there QUALITATIVE even in an autonomous eqn not all sol'ns must exist for all time, e.. $dy/dt = (1 + y)^2$ has sol'n $y = \tan t$, or $dy/dt = 1/(1 - y)$ has solns that go in finite time to 1 but no sol'n at 1 (open circle on phase line) SINK is an equilibrium point y_0 s.t. all sol'ns that start near y_0 are asymptotic to it as t increases SOURCE if a sol'n starts near a source y_0 then it is asymptotic to y_0 as t decreases NODE is an equilibrium point that is not a source or sink BIFURCATION occurs when a small change in a parameter to f results in a qualitative change in the long-term behavior of solutions BIFURCATION DIAGRAM for a one-parameter autonomous de family f_μ is a μy plot of the phase line for each value of μ with equilibrium points joined in curves PHASE PORTRAIT for a two-variable system $dR/dt = f(R, F), dF/dt = g(R, F)$ plots several solution curves in the RF phase plane, with arrows on the curves to indicate direction. Put solid dots for equilibrium points (constant solutions where the vector field vanishes). TRACE DETERMINANT PLANE for a 2 var system $Y' = AY$, if $T = trace(Y)$ and $D = det(Y)$, then 1. if $T = 0 \wedge D > 0$ the origin is a center 2. if $T > 0 \wedge D > T^2/4$ the origin is a spiral source 3. if $T < 0 \wedge D > T^2/4$ the origin is a spiral sink 4. if $T > 0 \wedge D < T^2/4$ the origin is a source 5. if $T < 0 \wedge D < T^2/4$ the origin is a sink 6. if $T > 0 \wedge D = T^2/4$ the origin is a source with repeated eigenvalues 7. if $T < 0 \wedge D = T^2/4$ the origin is a sink with repeated eigenvalues 8. if $D < 0$ the origin is a saddle LINEARIZATION to analyze the behavior of a system $dY/dt = F(Y)$ near an equilibrium point Y_0, transform the equilibrium point to the origin by setting $U = Y - Y_0$ and $dU/dt = F(Y) = F(U + Y_0)$. Then linearize by the Jacobian, approximating $dU/dt = DF|_0 U$, which can then be studied by linear methods. LINEARIZATION WORKS to describe the long-term behavior of solutions near an equilibrium point–so long as the linearized system is not a center and does not have 0 as an eigenvalue NULLCLINE of a system is the set of points where $dY_i/dt = 0$ for some i. By plotting nullclines for each component, we divide up the space into regions, which we can use to get a qualitative analysis of the system. SEPARATRIX of a saddle point is a solution curve along an eigenvector at the saddle point. A stable separatrix goes towards the saddle, an unstable separatrix goes away SADDLE CONNECTION is when (in a nonlinear system) a stable and unstable separatrix at a saddle point join together POINCARE RETURN MAP for a system with a forcing term that is a function of t with period T, is a function $f : R^n \to R^n, Y(t_0) \mapsto Y(t_T)$. By plotting the iterates of Y_0 under f, a picture of the chaotic system emerges. POLE of a rational function $G(s)/H(s)$ where G and H have no common factors is a value of s for which $s = 0$. In $d^2y/dt^2 + pdy/dt + qy = f(t)$, the poles of $\mathcal{L}[y]$ are analogous to the eigenvalues of the homogeneous equation. Some of the poles are those eigenvalues, and other poles correspond to the forcing term. THEOREM all poles of $\mathcal{L}(y)$ in the left complex half-plane implies stability (sol'ns tend to 0), and one or more poles in the right half-plane imply instability (sol'ns tending to ∞). EULERS METHOD $y_{k+1} = y_k + f(t_k, y_k)\Delta t$. Euler's method is a first order technique, i.e. the error $e \approx c\Delta t$ RUNGE KUTTA $y_{k+1} = y_k + (m_k + 2n_k + 3q_k + p_k)\Delta t/6$ where $m_k = f(t_k, y_k)$, $\tilde{y}_k = y_k + m_k\Delta t/2$, $\tilde{t} = t + \Delta t/2$, $n_k = f(\tilde{t}, \tilde{y}_k)$, $\hat{y} = y_k + n_k\Delta t/2$, $q_k = f(\tilde{t}, \hat{y}_k)$, $\bar{y}_k = y_k + q_k\Delta t$, $p_k = f(t_{k+1}, \bar{y}_k)$. Runge-kutta is a fourth order technique, i.e. the error $e \approx c(\Delta t)^4$ EFFECT OF FINITE ARITHMETIC means that in practice, R-K has error more like $e \approx c_1(\Delta t)^4 + c_d/\Delta t$. To determine which Δt minimizes the error, try different Δt (halving) until the solution seems to converge to a value before diverging again. LOGISTIC DIFFERENCE EQUATION $P_{n+1} = kP_n(1 - P_n)$. The function is $L_k(x) = kx(1 - x)$. ORBIT of a a seed P_0 under a function F is the sequence of iterates P_0, $F(P_0)$, $F^2(P_0) = F(F(P_0))$,$F^3(P_0) = F(F(F(P_0)))$, ... FIXED POINT of F is a point x where $F(x) = x$. Graphically it is where F intersects $y = x$. an orbit is "eventually fixed" if it eventually reaches a fixed point PERIODIC ORBIT also called a cycle; a point where $F^n(x) = F(F(...F(x)...)) = x$. n is the period of the orbit. an orbit is "eventually periodic" if it eventually enters a cycle. each point in the cycle is a fixed point of F^n DISPLAYING ORBITS GRAPHICALLY can be done as a simple line graph against the iteration number, or it can be done as a histogram, showing how often the orbit enters a given interval FINDING CYCLES can be done by solving $F^n(x) = x$. this can also be done graphically by graphing F^n and finding where it intersects $y = x$ (for a polynomial the fixed points factor out which can help) GRAPHICAL ITERATION draw the graph of F and the graph $y = x$. start at (x_0, x_0). When on the line $y = x$, trace vertically until reaching F. When on F, trace horizontally until reaching $y = x$. the resulting diagram is a "web diagram" ATTRACTING FIXED POINT is a fixed point x_0 about which there is an interval such that every seed in the interval has an orbit tending to x_0. this occurs if $|F'(x_0)| < 1$ REPELLING FIXED POINT is a fixed point x_0 about which there is an interval such that every seed in the interval except x_0 eventually leaves it. this occurs if $|F'(x_0)| > 1$ NEUTRAL FIXED POINT is a fixed point x_0 where $F'(x_0) = \pm 1$. orbits can behave in a variety of ways near neutral fixed points CLASSIFYING PERIODIC POINTS can be done in the same way as fixed points, except for F^n instead of F. THEOREM if x_0 is a periodic point of order n, then $(F^n)'(x_0) = F'(x_{n-1})F'(x_{n-1})...F'(x_0)$. thus $(F^n)'(x_k)$ is the same at all points on the cycle BIFURCATIONS of dynamical systems occur when fixed points and cycles appear or disappear, or change in type, as a parameter of the system varies TANGENT BIFURCATION also called a saddle node bifurcation, occurs when a fixed point or cycle disappears due to the graph of F^n becoming tangent to and then losing contact with $y = x$ PERIOD DOUBLING BIFURCATION occurs when a cycle changes from attracting to repelling or vice versa, with its derivative passing through -1, and that is accompanied by the birth of a new cycle having twice the original period CHOPPING FUNCTION $T(x) = 10x$ mod 1 for $0 \leq x < 1$ simply extracts and discards the first digit of x. its iterates are chaotic because each digit of x eventually becomes the most significant at some point. CHAOTIC periodic points of T form a dense subset of $[0, 1)$, there is a single orbit that forms a dense subset of $[0, 1)$, and the iterates of T are sensitively dependent on initial conditions LORENZ SYSTEM $dx/dt = \sigma(y - x)$, $dy/dt = \rho x - y - xz$, $dz/dt = -\beta z + xy$ exhibits chaotic behavior sensitive to initial conditions with infinitely many different qualitative behaviors STRANGE ATTRACTOR as in the Lorenz system is an attractor which behaves chaotically LORENZ TEMPLATE looks like a mask and shows the general appearance that solutions take. they loop around and repeatedly pass through an approximately line-shaped area. a Poincare return map can be constructed for the position on this line they return to, and its iterates constitute a discrete dynamic system.